1.5 ANALYZING GRAPHS OF FUNCTIONS

What You Should Learn

- Use the Vertical Line Test for functions.
- Find the zeros of functions.
- Determine intervals on which functions are increasing or decreasing and determine relative maximum and relative minimum values of functions.
- Determine the average rate of change of a function.
- Identify even and odd functions.

The Graph of a Function

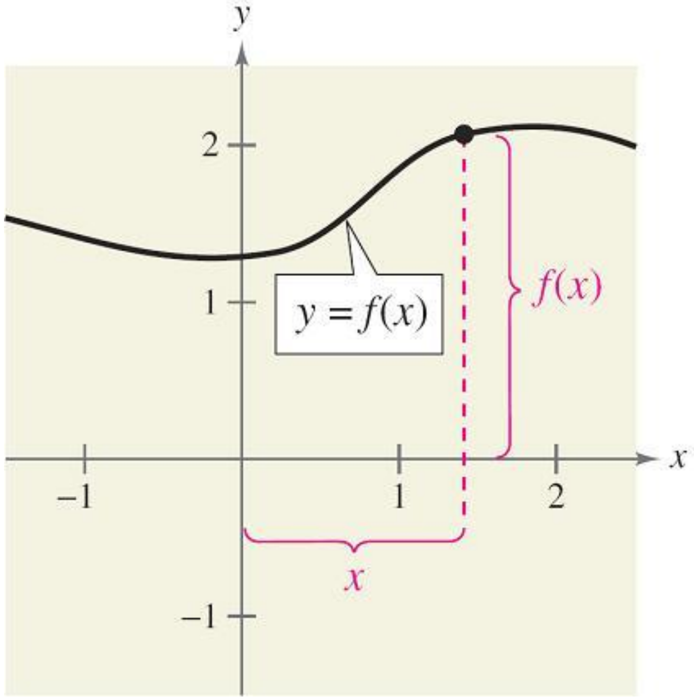
The Graph of a Function

The **graph of a function** *f* : the collection of ordered pairs (x, f(x)) such that x is in the domain of f.

The Graph of a Function

x = the directed distance from the y-axis

$y = f(x)$ = the directed distance from the x-axis



Example 1 – *Finding the Domain and Range of a Function*

Use the graph of the function *f*, shown in Figure 1.53, to find

(a) the domain of *f*,

(b) the function values *f*(–1) and *f*(2)

(c) the range of *f*.

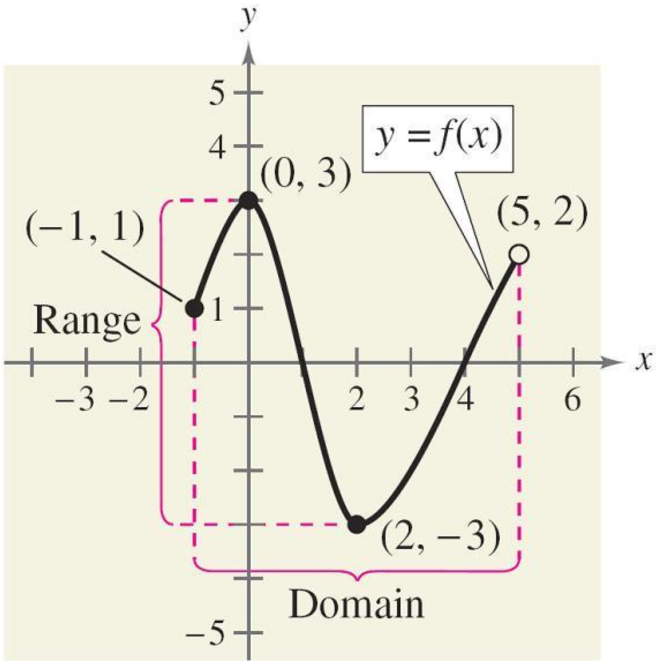


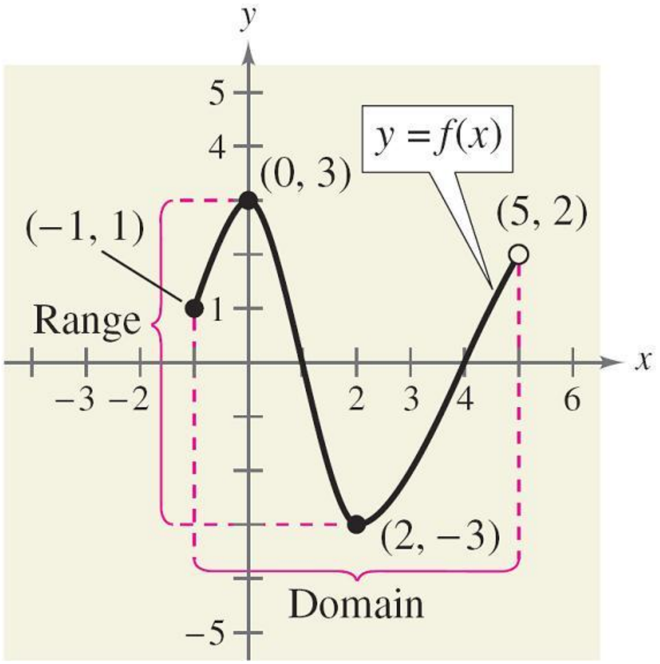
Figure 1.53

Example 1 – *Solution*

a. the domain of *f* is all *x* in the interval [–1, 5).

b. *f*(2) = –3.

c. the range of *f* is the interval [–3, 3].



The Graph of a Function

Vertical Line Test for Functions

A set of points in a coordinate plane is the graph of y as a function of x if and only if no *vertical* line intersects the graph at more than one point.

Zeros of a Function

Zeros of a Function

If the graph of a function of x has an x-intercept at $(a, 0)$, then a is a **zero** of the function.

Zeros of a Function

The **zeros of a function** f of x are the x-values for which $f(x) = 0$.

Example 3 – *Finding the Zeros of a Function*

Find the zeros of each function.

a. $f(x) = 3x^2 + x - 10$ **b.** $g(x) = \sqrt{10 - x^2}$ **c.** $h(t) = \dfrac{2t - 3}{t + 5}$

Solution:

To find the zeros of a function, set the function equal to zero and solve for the independent variable.

a. $3x^2 + x - 10 = 0$ — Set $f(x)$ equal to 0.

$(3x - 5)(x + 2) = 0$ — Factor.

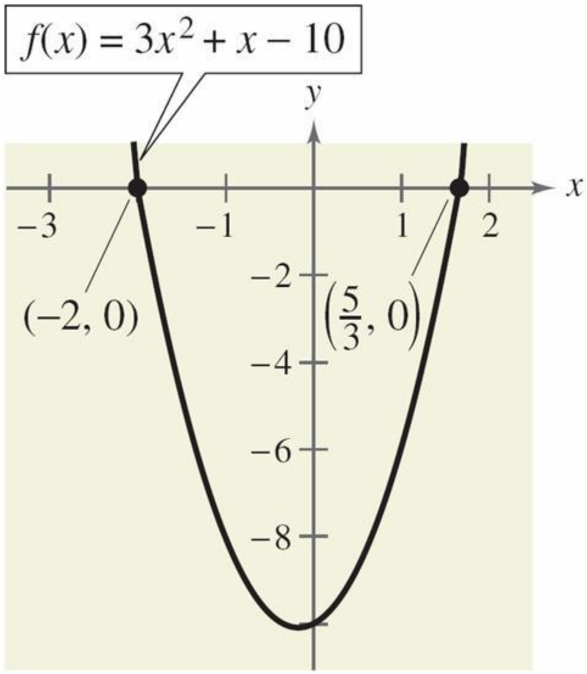
$3x - 5 = 0 \Rightarrow x = \dfrac{5}{3}$ — Set 1st factor equal to 0.

$x + 2 = 0 \Rightarrow x = -2$ — Set 2nd factor equal to 0.

Example 3 – *Solution*

cont'd

The zeros of *f* are $x = \frac{5}{3}$ and $x = -2$. In Figure 1.55, note that the graph of *f* has $(\frac{5}{3}, 0)$ and $(-2, 0)$ as its *x*-intercepts.



Zeros of *f*: $x = -2$, $x = \frac{5}{3}$

Figure 1.55

Example 3 – *Solution*

cont'd

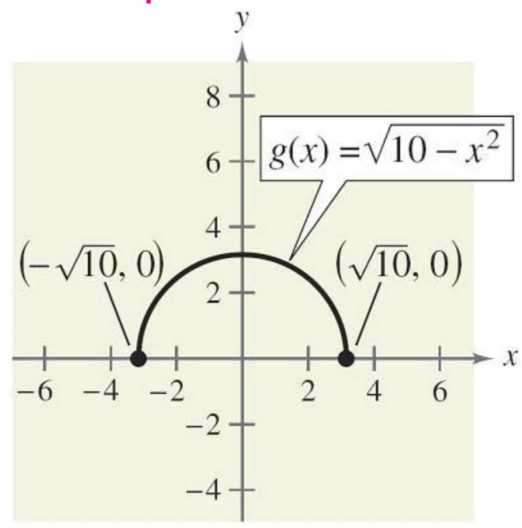
b. $\sqrt{10 - x^2} = 0$ — Set $g(x)$ equal to 0.

$10 - x^2 = 0$ — Square each side.

$10 = x^2$ — Add x^2 to each side.

$\pm\sqrt{10} = x$ — Extract square roots.

The zeros of g are $x = -\sqrt{10}$ and $x = \sqrt{10}$. In Figure 1.56, note that the graph of g has $(-\sqrt{10}, 0)$ and $(\sqrt{10}, 0)$ as its x-intercepts.



Zeros of g: $x = \pm\sqrt{10}$

Figure 1.56

Example 3 – *Solution*

cont'd

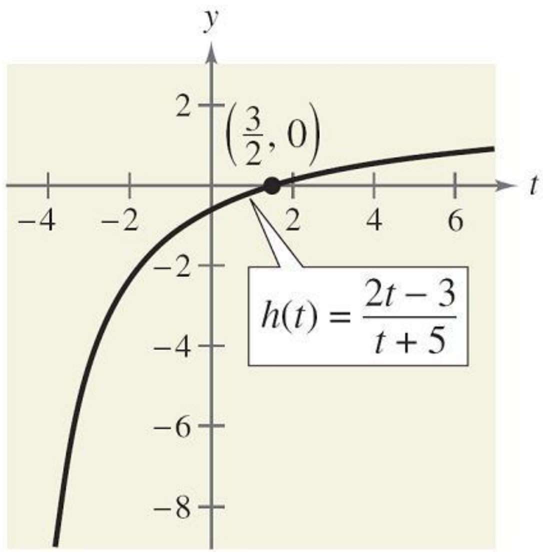
c. $\dfrac{2t - 3}{t + 5} = 0$ Set $h(t)$ equal to 0.

$2t - 3 = 0$ Multiply each side by $t + 5$

$2t = 3$ Add 3 to each side.

$t = \dfrac{3}{2}$ Divide each side by 2.

The zero of h is $t = \frac{3}{2}$. In Figure 1.57, note that the graph of h has $(\frac{3}{2}, 0)$ as its t-intercept.

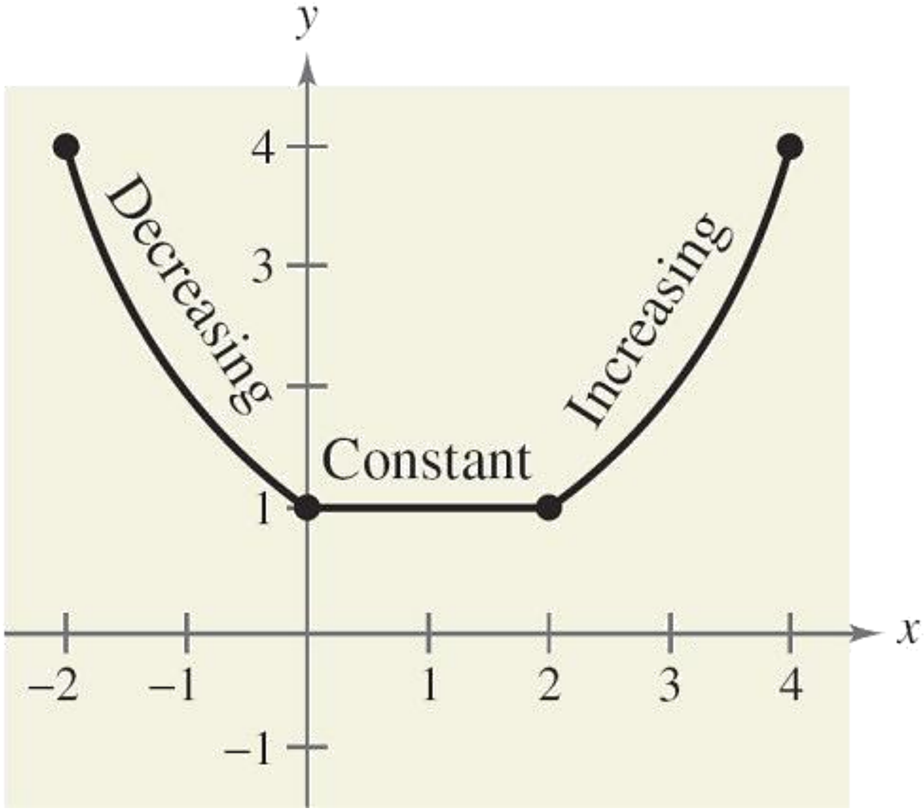


Zero of h: $t = \frac{3}{2}$

Figure 1.57

Increasing and Decreasing Functions

Increasing and Decreasing Functions



Increasing and Decreasing Functions

Increasing, Decreasing, and Constant Functions

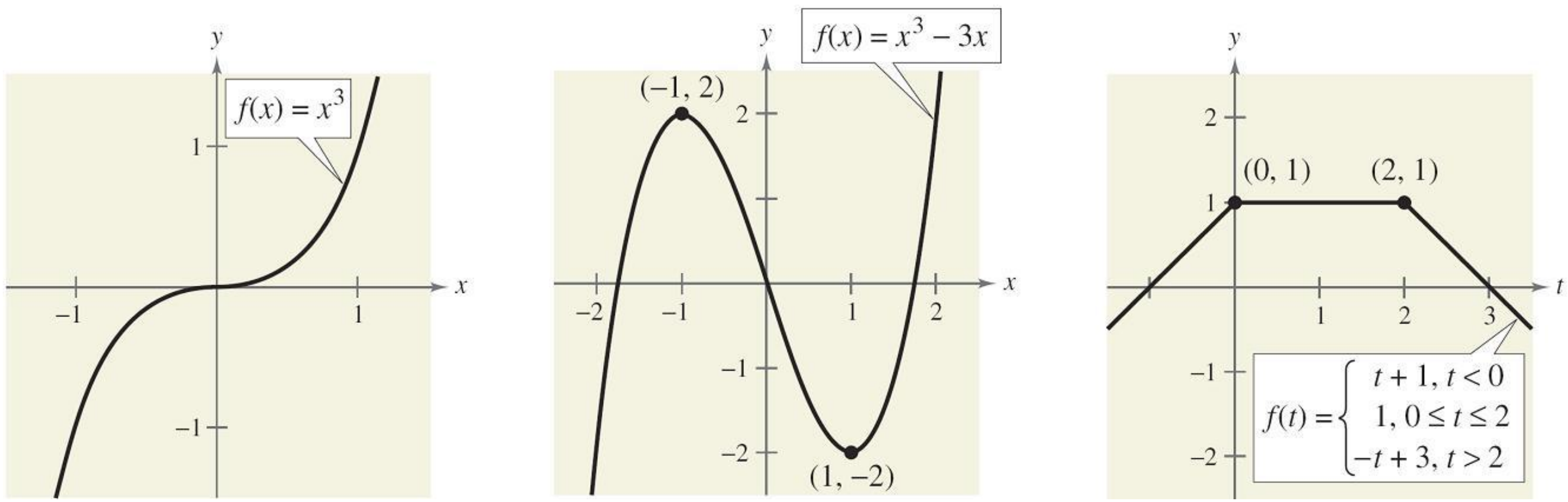
A function f is **increasing** on an interval if, for any x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) < f(x_2)$.

A function f is **decreasing** on an interval if, for any x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) > f(x_2)$.

A function f is **constant** on an interval if, for any x_1 and x_2 in the interval, $f(x_1) = f(x_2)$.

Example 4 – *Increasing and Decreasing Functions*

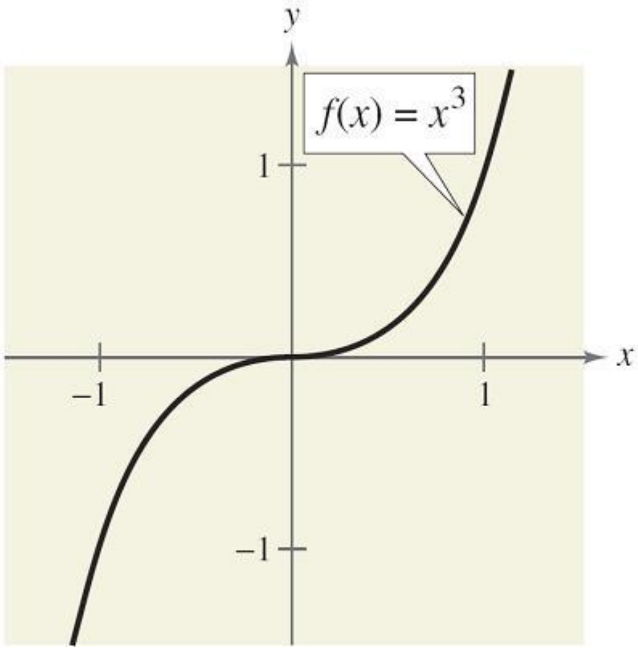
Describe the increasing or decreasing behavior of each function.



(a) (b) (c)

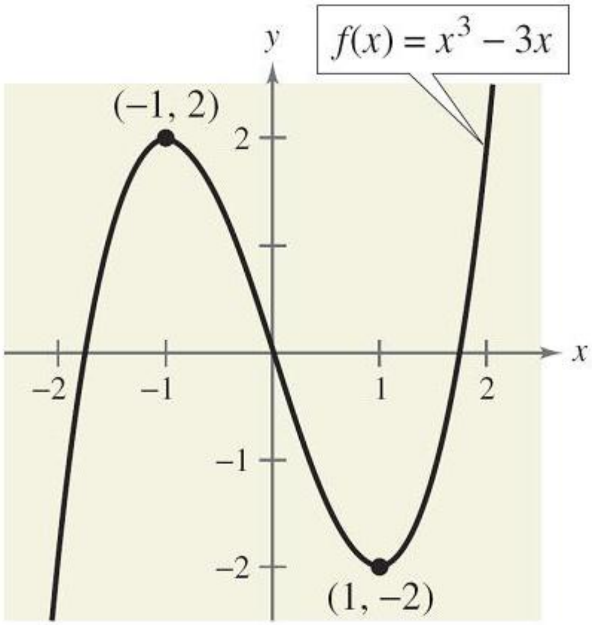
Example 4 – *Solution*

a. This function is increasing over the entire real line.



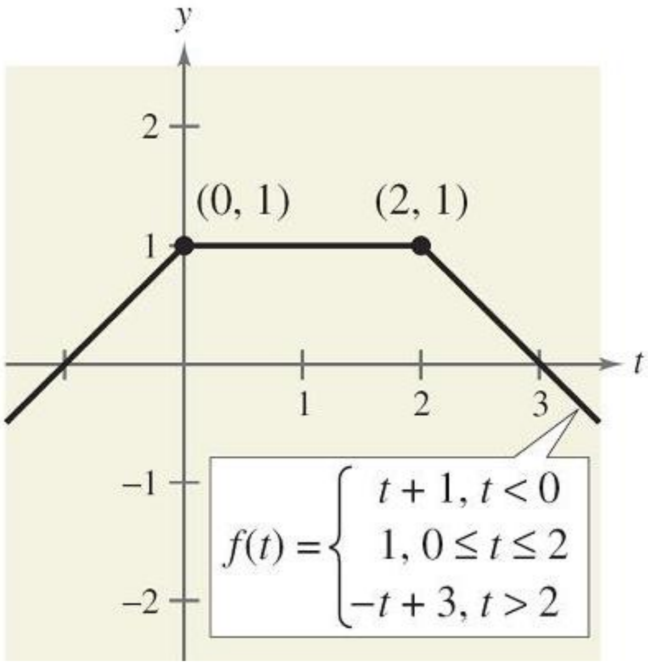
Example 4 – *Solution*

b. This function is increasing on the interval $(-\infty, -1)$, decreasing on the interval $(-1, 1)$ and increasing on the interval $(1, \infty)$



Example 4 – *Solution*

c. This function is increasing on the interval $(-\infty, 0)$, constant on the interval (0, 2), and decreasing on the interval $(2, \infty)$.



Increasing and Decreasing Functions

To help you decide whether a function is increasing, decreasing, or constant on an interval, you can evaluate the function for several values of *x*.

However, calculus is needed to determine, for certain, all intervals on which a function is increasing, decreasing, or constant.

Increasing and Decreasing Functions

Definitions of Relative Minimum and Relative Maximum

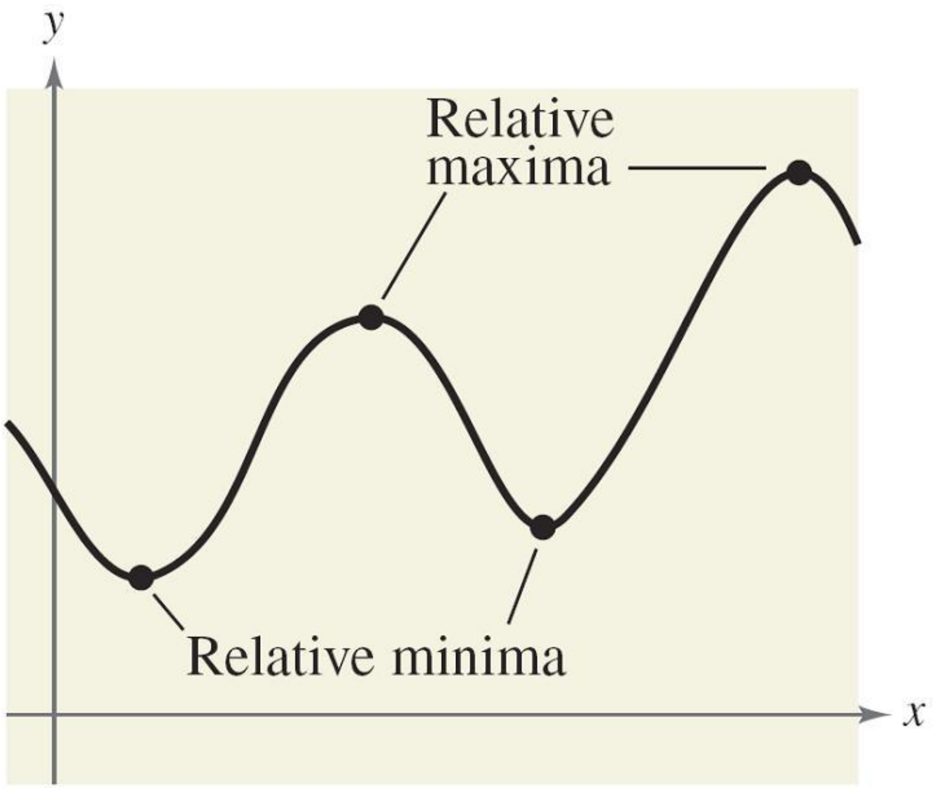
A function value $f(a)$ is called a **relative minimum** of f if there exists an interval (x_1, x_2) that contains a such that

$$x_1 < x < x_2 \quad \text{implies} \quad f(a) \le f(x).$$

A function value $f(a)$ is called a **relative maximum** of f if there exists an interval (x_1, x_2) that contains a such that

$$x_1 < x < x_2 \quad \text{implies} \quad f(a) \ge f(x).$$

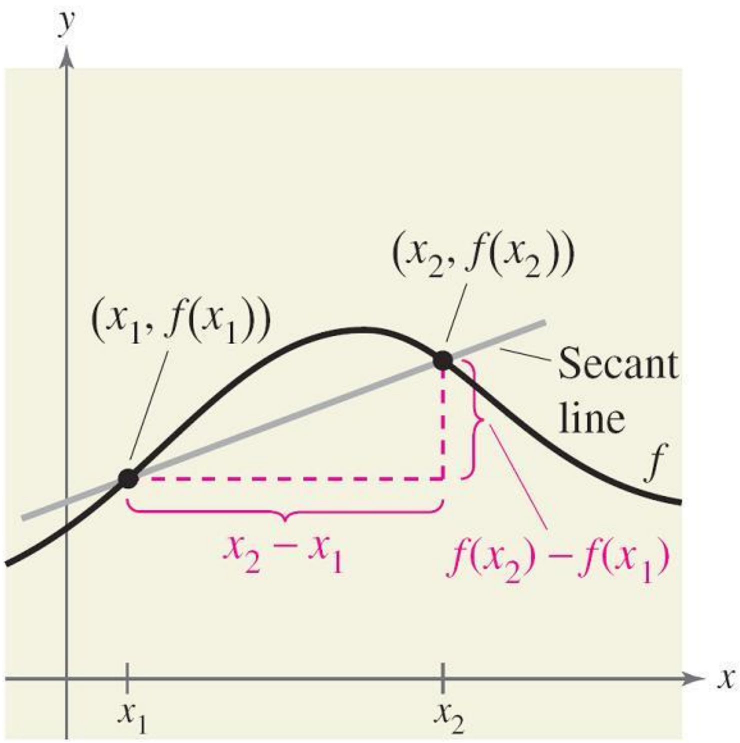
Increasing and Decreasing Functions



Average Rate of Change

Average Rate of Change

the **average rate of change** between any two points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ is the slope of the line through the two points.



Average Rate of Change

The line through the two points is called the **secant line,** and the slope of this line is denoted as m_{sec}.

Average rate of change of f from x_1 to x_2 $= \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

$$= \frac{\text{change in } y}{\text{change in } x}$$

$$= m_{sec}$$

Example 6 – *Average Rate of Change of a Function*

Find the average rates of change of $f(x) = x^3 - 3x$

(a) from $x_1 = -2$ to $x_2 = 0$ and

(b) from $x_1 = 0$ to $x_2 = 1$ (see Figure 1.63).

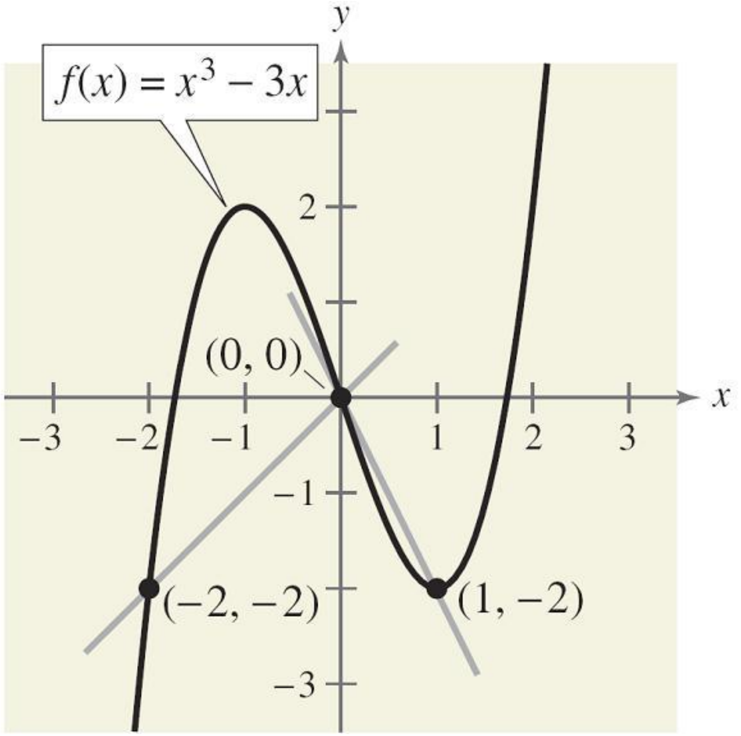


Figure 1.63

Example 6(a) – *Solution*

The average rate of change of f from $x_1 = -2$ to $x_2 = 0$ is

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(0) - f(-2)}{0 - (-2)}$$

$$= \frac{0 - (-2)}{2}$$

$$= 1.$$

Secant line has positive slope.

Example 6(b) – *Solution*

cont'd

The average rate of change of *f* from $x_1 = 0$ to $x_2 = 1$ is

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(1) - f(0)}{1 - 0}$$

$$= \frac{-2 - 0}{1}$$

$$= -2.$$

Secant line has negative slope.

Even and Odd Functions

Even and Odd Functions

Tests for Even and Odd Functions

A function $y = f(x)$ is **even** if, for each x in the domain of f,

$$f(-x) = f(x).$$

A function $y = f(x)$ is **odd** if, for each x in the domain of f,

$$f(-x) = -f(x).$$

Example 8 – *Even and Odd Functions*

a. The function $g(x) = x^3 - x$ is odd because $g(-x) = -g(x)$, as follows.

$$g(-x) = (-x)^3 - (-x)$$ Substitute $-x$ for x.

$$= -x^3 + x$$ Simplify.

$$= -(x^3 - x)$$ Distributive Property

$$= -g(x)$$ Test for odd function

Example 8 – *Even and Odd Functions*

b. The function $h(x) = x^2 + 1$ is even because $h(-x) = h(x)$, as follows.

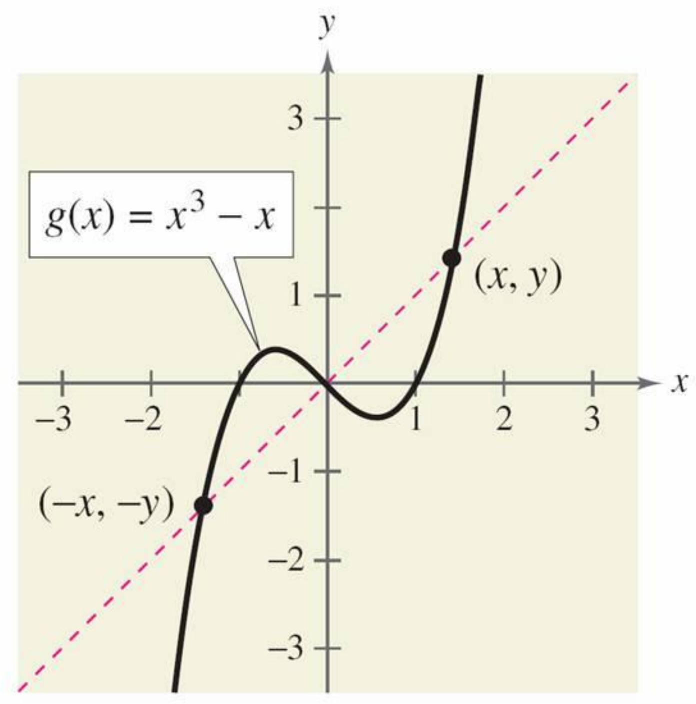
$h(-x) = (-x)^2 + 1$ Substitute $-x$ for x.

$= x^2 + 1$ Simplify.

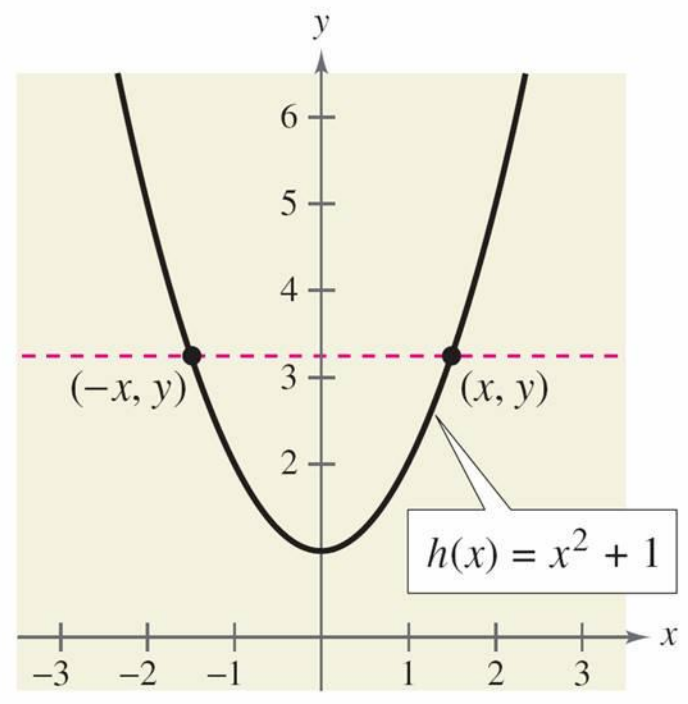
$= h(x)$ Test for even function

Example 8 – *Even and Odd Functions* cont'd

The graphs and symmetry of these two functions are shown in Figure 1.64.



(a) Symmetric to origin: Odd Function



(b) Symmetric to *y*-axis: Even Function